

# MATHEMATICAL MODELS OF SPONTANEOUS SYMMETRY BREAKING

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**Abstract.** The Higgs mechanism of mass generation is the main ingredient in the contemporary Standard Model and its various generalizations. However, there is no comprehensive theory of spontaneous symmetry breaking. We summarize the relevant mathematical results characterizing spontaneous symmetry breaking in algebraic quantum theory, axiomatic quantum field theory, group theory, and classical gauge theory.

The Higgs mechanism of mass generation is the main ingredient in the contemporary Standard Model of high energy physics and its various generalizations. The key point of this mechanism is interaction of particles and fields with a certain multiplet of Higgs fields associated to some gauge symmetry group and possessing nonzero vacuum expectations. The latter is treated as spontaneous symmetry breaking. The Higgs mechanism has been extended to different gauge theories including SUSY gauge theory [12, 40, 43, 56], noncommutative field theory [19, 39, 44], gravitation theory [1, 13, 31, 33, 35], the gauge Landau–Ginzburg–Higgs theory [3, 23]. In quantum field theory, spontaneous symmetry breaking phenomena imply that a physical vacuum is not the bare Fock one, but it possesses nonzero physical characteristics and, thus, can interact with particles and fields.

At present, there is no comprehensive theory of spontaneous symmetry breaking. In order to attract attention to this problem, let us summarize the relevant mathematical results characterizing spontaneous symmetry breaking in algebraic quantum theory, axiomatic quantum field theory, group theory, and classical gauge theory.

## 1 Algebraic quantum theory

In algebraic quantum theory, a quantum system is characterized by a topological involutive algebra  $A$  and positive continuous forms  $f$  on  $A$ . If  $A$  is a Banach algebra admitting an approximate identity (in particular, a  $C^*$ -algebra), the well-known Gelfand–Naimark–Segal (GNS) representation theorem associates to any positive continuous form on an algebra  $A$  its cyclic representation by bounded (continuous) operators in a Hilbert space [18]. There exist different extensions of this GNS representation theorem [20]. For instance, axiomatic

quantum field theory (QFT) deals with unnormed unital topological involutive algebras. Given such an algebra  $A$ , the GNS representation theorem associates to a state  $f$  of  $A$  a strongly cyclic Hermitian representation  $(\pi_f, \theta_f)$  of  $A$  by an  $Op^*$ -algebra  $\pi(A)$  of unbounded operators in a Hilbert space such that  $f(a) = \langle \pi(a)\theta_f | \theta_f \rangle$ ,  $a \in A$  [25, 51].

Given a topological involutive algebra  $A$  and its representation  $\pi$  in a Hilbert space  $E$ , one speaks about spontaneous symmetry breaking if there are automorphisms of  $A$  which do not admit a unitary representation in  $E$ .

Recall that an automorphism  $\rho$  of  $A$  possesses a unitary representation in  $E$  if there exists a unitary operator  $U_\rho$  in  $E$  such that

$$\pi(\rho(a)) = U_\rho \pi(a) U_\rho^{-1}, \quad a \in A. \quad (1)$$

A problem is that such a representation is never unique. Namely, let  $U$  and  $U'$  be arbitrary unitary elements of the commutant  $\pi(A)'$  of  $\pi(A)$ . Then  $UU_\rho U'$  also provides a unitary representation of  $\rho$ . For instance, one can always choose phase multipliers  $U = \exp(i\alpha)\mathbf{1} \in U(1)$ . A consequence of this ambiguity is the following.

Let  $G$  be a group of automorphisms of an algebra  $A$  whose elements  $g \in G$  admit unitary representations  $U_g$  (1). The set of operators  $U_g$ ,  $g \in G$ , however need not be a group. In general, we have

$$U_g U_{g'} = U(g, g') U_{gg'} U'(g, g'), \quad U(g, g'), U'(g, g') \in \pi(A)'.$$

If all  $U(g, g')$  are phase multipliers, one says that the unitary operators  $U_g$ ,  $g \in G$ , form a projective representation  $U(G)$ :

$$U_g U_{g'} = k(g, g') U_{gg'}, \quad g, g' \in G,$$

of a group  $G$  [11, 55]. In this case, the set  $U(1) \times U(G)$  becomes a group which is a central  $U(1)$ -extension

$$\mathbf{1} \longrightarrow U(1) \longrightarrow U(1) \times U(G) \longrightarrow G \longrightarrow \mathbf{1} \quad (2)$$

of a group  $G$ . Accordingly, the projective representation  $\pi(G)$  of  $G$  is a splitting of the exact sequence (2). It is characterized by  $U(1)$ -multipliers  $k(g, g')$  which form a two-cocycle

$$k(\mathbf{1}, g) = k(g, \mathbf{1}) = \mathbf{1}, \quad k(g_1, g_2 g_3) k(g_2, g_3) = k(g_1, g_2) k(g_1 g_2, g_3) \quad (3)$$

of the cochain complex of  $G$  with coefficients in  $U(1)$  [20, 37]. A different splitting of the exact sequence (2) yields a different projective representation  $U'(G)$  of  $G$  whose multipliers  $k'(g, g')$  form a cocycle equivalent to the cocycle (3). If this cocycle is a coboundary, there exists a splitting of the extension (2) which provides a unitary representation of a group  $G$  of automorphisms of an algebra  $A$  in  $E$ .

For instance, let  $B(E)$  be the  $C^*$ -algebra of all bounded operators in a Hilbert space  $E$ . Any automorphisms of  $B(E)$  is inner and, consequently, possesses a unitary representation in  $E$ . Since the commutant of  $B(E)$  reduces to scalars, the group of automorphisms of  $B(E)$  admits a projective representation in  $E$ , but it need not be unitary.

One can say something if  $A$  is a  $C^*$ -algebra and its GNS representations are considered [10, 18, 20].

**Theorem 1.** Let  $f$  be a state of a  $C^*$ -algebra  $A$  and  $(\pi_f, \theta_f, E_f)$  the corresponding GNS representation of  $A$ . An automorphism  $\rho$  of  $A$  defines a state

$$(\rho f)(a) = f(\rho(a)), \quad a \in A, \quad (4)$$

of  $A$  such that the carrier space  $E_{\rho f}$  of the corresponding GNS representation  $\pi_{\rho f}$  is isomorphic to  $E_f$ .

It follows that the representations  $\pi_{\rho f}$  can be given by operators  $\pi_{\rho f}(a) = \pi_f(\rho(a))$  in the carrier space  $E_f$  of the representation  $\pi_f$ , but these representations fail to be equivalent, unless an automorphism  $\rho$  possesses a unitary representation (1) in  $E_f$ .

**Theorem 2.** If a state  $f$  of a  $C^*$ -algebra  $A$  is stationary

$$f(\rho(a)) = f(a), \quad a \in A, \quad (5)$$

with respect to an automorphism  $\rho$  of  $A$ , there exists a unique unitary representation  $U_\rho$  (4) of  $\rho$  in  $E_f$  such that

$$U_\rho \theta_f = \theta_f. \quad (6)$$

A topological group  $G$  is called a strongly (resp. uniformly) continuous group of automorphisms of a  $C^*$ -algebra  $A$  if there is its continuous monomorphism to the group  $\text{Aut}(A)$  of automorphisms of  $A$  provided with the strong (resp. normed) operator topology, and if its action on  $A$  is separately continuous. An infinitesimal generator  $\delta$  of a strongly continuous one-parameter group  $G(\mathbb{R})$  of automorphisms of a  $C^*$ -algebra  $A$  is an unbounded derivation of  $A$  [10, 45]. This derivation is bounded iff a group  $G(\mathbb{R})$  is uniformly continuous.

**Theorem 3.** If a one-parameter group  $G(\mathbb{R})$  of automorphisms of a  $C^*$ -algebra  $A$  is uniformly continuous, any representation  $\pi$  of  $A$  in a Hilbert space  $E$  yields the unitary representation of the group  $G(\mathbb{R})$  in  $E$ :

$$\pi(g(t)) = \exp(-it\mathcal{H}), \quad \pi(\delta(a)) = -i[\mathcal{H}, \pi(a)], \quad a \in A, \quad (7)$$

where  $\mathcal{H} \in \pi(A)''$  is a bounded self-adjoint operator in  $E$  [10, 20].

A problem is that a  $C^*$ -algebra need not admit nonzero bounded derivations. For instance, no commutative  $C^*$ -algebra possesses bounded derivations. Given a strongly continuous one-parameter group  $G(\mathbb{R})$  of automorphisms of a  $C^*$ -algebra  $A$ , a representation of  $A$  need not imply a unitary representation (7) of this group, unless the following sufficient condition holds.

**Theorem 4.** Let  $f$  be a state of a  $C^*$ -algebra  $A$  such that

$$|f(\delta(a))| \leq \lambda[f(a^*a) + f(aa^*)]^{1/2}$$

for all  $a \in A$  and some positive number  $\lambda$ , and let  $(\pi_f, \theta_f)$  be the corresponding GNS representation of  $A$  in a Hilbert space  $E_f$ . Then there exist a self-adjoint operator  $\mathcal{H}$  on a domain  $D \subset \pi_f(A)\theta_f$  in  $E_f$  and a strongly continuous unitary representation (7) of  $G(\mathbb{R})$  in  $E_f$ .

For instance, any strongly continuous one-parameter group of automorphisms of a  $C^*$ -algebra  $B(E)$  possesses a unitary representation in  $E$ .

**Theorem 5.** Let  $G$  be a strongly continuous group of automorphisms of a  $C^*$ -algebra  $A$ , and a state  $f$  of  $A$  be stationary for  $G$ . Then there exists a unique unitary representation of  $G$  in  $E_f$  whose operators obey the equality (6).

## 2 Axiomatic QFT

In axiomatic QFT, the spontaneous symmetry breaking phenomenon is described by the Goldstone theorem [8].

There are two main algebraic formulations of QFT. In the framework of the first one, called local QFT, one associates to a certain class of subsets of a Minkowski space a net of von Neumann,  $C^*$ - or  $Op^*$ -algebras which obey certain axioms [4, 24, 25]. Its inductive limit is called either a global algebra (in the case of von Neumann algebras) or a quasilocal algebra (for a net of  $C^*$ -algebras).

In a different formulation of algebraic QFT, quantum field algebras are tensor algebras. Let  $Q$  be a nuclear space. Let us consider the direct limit

$$A_Q = \hat{\otimes} Q = \mathbb{C} \oplus Q \oplus Q \hat{\otimes} Q \oplus \cdots Q^{\hat{\otimes} n} \oplus \cdots \quad (8)$$

of the vector spaces

$$\hat{\otimes}^{\leq n} Q = \mathbb{C} \oplus Q \oplus Q \hat{\otimes} Q \cdots \oplus Q^{\hat{\otimes} n},$$

where  $\hat{\otimes}$  is the topological tensor product with respect to Grothendieck's topology. One can show that, provided with the inductive limit topology, the tensor algebra  $A_Q$  (8) is a unital

nuclear barreled b-algebra [6, 26]. Therefore, one can apply to it the GNS representation theorem. Namely, if  $f$  is a positive continuous form on  $A$ , there exists a unique cyclic representation  $\pi_f$  of  $A$  in a Hilbert space by operators on a common invariant domain  $D$  [26]. This domain can be topologized to conform a rigged Hilbert space such that all the operators representing  $A$  are continuous on  $D$ . Herewith, a linear form  $f$  on  $A_Q$  is continuous iff the restriction of  $f$  to each  $\widehat{\otimes}^{\leq n} Q$  is so [54].

In algebraic QFT, one usually choose  $Q$  the Schwartz space of functions of rapid decrease. For the sake of simplicity, we here restrict our consideration to real scalar fields. One associates to them the Borchers algebra

$$A = \mathbb{R} \oplus RS^4 \oplus RS^8 \oplus \dots, \quad (9)$$

where  $RS^{4k}$  is the nuclear space of smooth real functions of rapid decrease on  $\mathbb{R}^{4k}$  [9, 25]. It is the real subspace of the space  $S(\mathbb{R}^{4k})$  of smooth complex functions of rapid decrease on  $\mathbb{R}^{4k}$ . Its topological dual is the space  $S'(\mathbb{R}^{4k})$  of tempered distributions (generalized functions). Since the subset  $\overset{k}{\otimes} S(\mathbb{R}^4)$  is dense in  $S(\mathbb{R}^{4k})$ , we henceforth identify  $A$  with the tensor algebra  $A_{RS^4}$  (8). Then any continuous positive form on the Borchers algebra  $A$  (9) is represented by a collection of tempered distributions  $\{W_k \in S'(\mathbb{R}^{4k})\}$  such that

$$f(\psi_k) = \int W_k(x_1, \dots, x_k) \psi_k(x_1, \dots, x_k) d^4x_1 \cdots d^4x_k, \quad \psi_k \in RS^{4k}.$$

For instance, the states of scalar quantum fields in the Minkowski space  $\mathbb{R}^4$  are described by the Wightman functions  $W_n \in S'(\mathbb{R}^{4k})$  in the Minkowski space which obey the Garding–Wightman axioms of axiomatic QFT [8, 57, 58]. Let us mention the Poincaré covariance axiom, the condition of the existence and uniqueness of a vacuum  $\theta_0$ , and the spectrum condition. They imply that: (i) the carrier Hilbert space  $E_W$  of Wightman quantum fields admits a unitary representation of the Poincaré group, (ii) the space  $E_W$  contains a unique (up to scalar multiplications) vector  $\psi_0$ , called the vacuum vector, invariant under Poincaré transformations, (iii) the spectrum of the energy-momentum operator lies in the closed positive light cone. Let  $G$  be a connected Lie group of internal symmetries (automorphisms of the Borchers algebra  $A$  over  $\text{Id } \mathbb{R}^4$ ) whose infinitesimal generators are given by conserved currents  $j_\mu^k$ . One can show the following [8].

**Theorem 6.** A group  $G$  of internal symmetries possesses a unitary representation in  $E_W$  iff the Wightman functions are  $G$ -invariant.

**Theorem 7.** A group  $G$  of internal symmetries admits a unitary representation if a strong spectrum condition holds, i.e., there exists a mass gap.

As a consequence, we come to the above mentioned Goldstone theorem.

**Theorem 8.** If there is a group  $G$  of internal symmetries which are spontaneously broken, there exist elements  $\phi \in E_W$  of zero spin and mass such that  $\langle \phi | j_\mu^k \psi_0 \rangle \neq 0$  for some generators of  $G$ .

These elements of unit norm are called Goldstone states. It is easily observed that, if a group  $G$  of spontaneously broken symmetries contains a subgroup of exact symmetries  $H$ , the Goldstone states carrier out a homogeneous representation of  $G$  isomorphic to the quotient  $G/H$ . This fact attracted great attention to such kind representations.

### 3 Nonlinear realizations of Lie algebras and superalgebras

In a general setting, given a Lie group  $G$  and its Lie subgroup  $H$ , one can construct an induced representation of a group  $G$  on a space of functions  $f$  on  $G$  taking values in a carrier space  $V$  of some representation of  $H$  such that  $f(gh) = h^{-1}f(g)$  for all  $h \in H$ ,  $g \in G$  [15, 36]. If  $G \rightarrow G/H$  is a trivial fiber bundle, there exists its global section whose values are representatives of elements of  $G/H$ . Given such a section  $s$ , the product  $G/H \times V$  can be provided with the particular induced representation

$$G \ni g : (\sigma, v) \mapsto (g\sigma, g_\sigma v), \quad g_\sigma = s(g\sigma)^{-1}gs(\sigma) \in H, \quad (10)$$

of  $G$ . If  $H$  is a Cartan subgroup of  $G$ , the well known nonlinear realization of  $G$  in a neighborhood of its unit [16, 30] exemplifies the induced representation (10). In fact, it is a representation of the Lie algebra of  $G$  around its origin as follows.

The Lie algebra  $\mathfrak{g}$  of a Lie group  $G$  containing a Cartan subgroup  $H$  is split into the sum  $\mathfrak{g} = \mathfrak{f} + \mathfrak{h}$  of the Lie algebra  $\mathfrak{h}$  of  $H$  and its supplement  $\mathfrak{f}$  obeying the commutation relations

$$[\mathfrak{f}, \mathfrak{f}] \subset \mathfrak{h}, \quad [\mathfrak{f}, \mathfrak{h}] \subset \mathfrak{f}.$$

In this case, there exists an open neighbourhood  $U$  of the unit of  $G$  such that any element  $g \in U$  is uniquely brought into the form

$$g = \exp(F) \exp(I), \quad F \in \mathfrak{f}, \quad I \in \mathfrak{h}.$$

Let  $U_G$  be an open neighbourhood of the unit of  $G$  such that  $U_G^2 \subset U$ , and let  $U_0$  be an open neighbourhood of the  $H$ -invariant center  $\sigma_0$  of the quotient  $G/H$  which consists of elements

$$\sigma = g\sigma_0 = \exp(F)\sigma_0, \quad g \in U_G.$$

Then there is a local section  $s(g\sigma_0) = \exp(F)$  of  $G \rightarrow G/H$  over  $U_0$ . With this local section, one can define the induced representation (10) of elements  $g \in U_G \subset G$  on  $U_0 \times V$  given by

the expressions

$$g \exp(F) = \exp(F') \exp(I'), \quad (11)$$

$$g : (\exp(F)\sigma_0, v) \mapsto (\exp(F')\sigma_0, \exp(I')v). \quad (12)$$

The corresponding representation of the Lie algebra  $\mathfrak{g}$  of  $G$  takes the following form. Let  $\{F_\alpha\}$ ,  $\{I_a\}$  be the bases for  $\mathfrak{f}$  and  $\mathfrak{h}$ , respectively. Their elements obey the commutation relations

$$[I_a, I_b] = c_{ab}^d I_d, \quad [F_\alpha, F_\beta] = c_{\alpha\beta}^d I_d, \quad [F_\alpha, I_b] = c_{\alpha b}^\beta F_\beta.$$

Then the relation (11) leads to the formulas

$$F_\alpha : F \mapsto F' = F_\alpha + \sum_{k=1} l_{2k} [\cdot \cdot \cdot [F_\alpha, F], F], \dots, F] - l_n \sum_{n=1} [\cdot \cdot \cdot [F, I'], I'], \dots, I'], \quad (13)$$

$$I' = \sum_{k=1} l_{2k-1} [\cdot \cdot \cdot [F_\alpha, F], F], \dots, F], \quad (14)$$

$$I_a : F \mapsto F' = 2 \sum_{k=1} l_{2k-1} [\cdot \cdot \cdot [I_a, F], F], \dots, F], \quad I' = I_a, \quad (15)$$

where coefficients  $l_n$ ,  $n = 1, \dots$ , are obtained from the recursion relation

$$\frac{n}{(n+1)!} = \sum_{i=1}^n \frac{l_i}{(n+1-i)!}. \quad (16)$$

Let  $U_F$  be an open subset of the origin of the vector space  $\mathfrak{f}$  such that the series (13) – (15) converge for all  $F \in U_F$ ,  $F_\alpha \in \mathfrak{f}$  and  $I_a \in \mathfrak{h}$ . Then the above mentioned nonlinear realization of the Lie algebra  $\mathfrak{g}$  in  $U_F \times V$  reads

$$F_\alpha : (F, v) \mapsto (F', I'v), \quad I_a : (F, v) \mapsto (F', I'v), \quad (17)$$

where  $F'$  and  $I'$  are given by the expressions (13) – (15). In physical models, the coefficients  $\sigma^\alpha$  of  $F = \sigma^\alpha F_\alpha$  are treated as Goldstone fields.

Nonlinear realizations of many groups especially in application to gravitation theory have been studied [27, 33, 35, 53]. Furthermore, SUSY gauge theory including supergravity is mainly developed as a Yang–Mills type theory with spontaneous breaking of supersymmetries [7, 41, 43, 56]. For instance, let us mention various superextensions of the pseudo-orthogonal and Poincaré Lie algebras [2, 17]. The nonlinear realization of a number of Lie superalgebras have been obtained [14, 29] in accordance with the following scheme.

Let  $G$  be a Lie supergroup in the category of  $G$ -supermanifolds [5], and let  $\widehat{H}$  be its Lie supersubgroup such that the even part  $\widehat{\mathfrak{h}}_0$  of its Lie superalgebra is a Cartan subalgebra of

the Lie algebra  $\widehat{\mathfrak{g}}_0$ . With  $F, F', F'' \in \widehat{\mathfrak{f}}_0$  and  $I', I'' \in \widehat{\mathfrak{h}}_0$ , we can repeat the relations (11), (13) – (15) as follows:

$$F'' \exp(F) = \exp(F') \exp(I'),$$

$$F' = F'' + \sum_{k=1} l_{2k} [\cdot, \cdot]_{2k} [F'', F], [F], \dots, [F] - l_n \sum_{n=1} [\cdot, \cdot]_n [F, I'], [I'], \dots, [I'], \quad (18)$$

$$I' = \sum_{k=1} l_{2k-1} [\cdot, \cdot]_{2k-1} [F'', F], [F], \dots, [F], \quad (19)$$

$$I'' \exp(F) = \exp(F') \exp(I'),$$

$$F' = 2 \sum_{k=1} f l_{2k-1} [\cdot, \cdot]_{2k-1} [I'', F], [F], \dots, [F], \quad I' = I'', \quad (20)$$

where coefficients  $l_n$ ,  $n = 1, \dots$ , are obtained from the formula (16). Let a superspace  $\widehat{V}$  carries out a linear representation of the Lie superalgebra  $\widehat{\mathfrak{h}}$ . Let  $\widehat{U}_F$  be an open subset of the supervector space  $\widehat{\mathfrak{f}}_0$  such that the series (18) – (20) converge for all  $F \in \widehat{U}_F$ ,  $F'' \in \widehat{\mathfrak{f}}_0$  and  $I'' \in \widehat{\mathfrak{h}}_0$ . Then we obtain the following nonlinear realization of the even Lie algebra  $\widehat{\mathfrak{g}}_0$  in  $\widehat{U}_F \times \widehat{V}$ :

$$F'' : (F, v) \mapsto (F', I'v), \quad I'' : (F, v) \mapsto (F', I'v),$$

where  $F'$  and  $I'$  are given by the expressions (18) – (20).

## 4 Classical gauge theory

If  $G$  is a real Lie group and  $H$  is its closed (and, consequently, Lie) subgroup, classical fields taking values in the quotient space  $G/H$  characterize spontaneous breaking phenomena in classical gauge theory. They are called Higgs fields.

In gauge theory on a principal bundle  $P \rightarrow X$  with a structure Lie group  $G$ , gauge potentials are identified to principal connections on  $P \rightarrow X$ . Being equivariant under the canonical action of  $G$  on  $P$ , these connections are represented by sections of the quotient  $C = J^1 P / G$ , of the first order jet manifold  $J^1 P$  of the principal bundle  $P \rightarrow X$ . It is an affine bundle coordinated by  $(x^\lambda, a_\lambda^r)$  such that, given a section  $A$  of  $C \rightarrow X$ , its components  $A_\lambda^r = a_\lambda^r \circ A$  are coefficients of the familiar local connection form [34], i.e., gauge potentials.

In gauge theory on a principal bundle  $P \rightarrow X$ , matter fields are represented by sections of an associated bundle

$$Y = (P \times V) / G, \quad (21)$$

where  $V$  is a vector space which the structure group  $G$  acts on, and the quotient (21) is defined by identification of the elements  $(p, v)$  and  $(pg, g^{-1}v)$  for all  $g \in G$ . Any principal connection  $A$  on  $P$  yields an associated linear connection on the associated bundle (21).



Given bundle coordinates  $(x^\mu, y^i)$  on  $Y$ , this connection takes the form

$$A = dx^\mu \otimes (\partial_\mu + A_\mu^r I_r^i \partial_i),$$

where  $I_r$  are generators of a representation of a group  $G$  in  $V$ .

Spontaneous symmetry breaking in classical gauge theory occurs if matter fields carrier out a representation only of some subgroup  $H$  of a group  $G$ . To describe these fields, one should assume that the structure group  $G$  of a principal bundle  $P \rightarrow X$  is reduced to its closed Lie subgroup  $H$ , i.e.,  $P$  contains an  $H$ -principal subbundle called a  $G$ -structure [22, 32, 42, 46, 50]. We have the composite bundle

$$P \xrightarrow{\pi_{P\Sigma}} P/H \longrightarrow X,$$

where

$$P_\Sigma = P \xrightarrow{\pi_{P\Sigma}} P/H \tag{22}$$

is a principal bundle with the structure group  $H$  and

$$\Sigma = P/H \xrightarrow{\pi_{\Sigma X}} X$$

is a  $P$ -associated fiber bundle with the typical fiber  $G/H$  which the structure group  $G$  acts on on the left.

**Theorem 9.** There is one-to-one correspondence between the reduced  $H$ -principal subbundles  $P^h$  of  $P$  and the global sections  $h$  of the quotient bundle  $P/H \rightarrow X$  [50, 52].

These sections are the above mentioned classical Higgs fields. Given such a section  $h$ , the corresponding reduced subbundle is the pull-back

$$P^h = h^* P_\Sigma = \pi_{P\Sigma}^{-1}(h(X)) \tag{23}$$

of the  $H$ -principal bundle (22) onto  $h(X) \subset \Sigma$ .

In general, there is a topological obstruction to the reduction of a structure group of a principal bundle to its subgroup. One usually refers to the following fact.

**Theorem 10.** Any fiber bundle whose typical fiber is diffeomorphic to an Euclidean space has a global section [52]. In particular, any structure group  $G$  of a principal bundle is always reducible to its maximal compact subgroup  $H$  since the quotient space  $G/H$  is homeomorphic to an Euclidean space.

For instance, this is the case of the groups  $G = GL(n, \mathbb{C})$ ,  $H = U(n)$  and  $G = GL(n, \mathbb{R})$ ,  $H = O(n)$ . In the last case, the associated Higgs field is a Riemannian metric on  $X$ .

It should be emphasized that different  $H$ -principal subbundles  $P^h$  and  $P^{h'}$  of a  $G$ -principal bundle  $P$  need not be isomorphic to each other in general. They are isomorphic over  $X$  iff there is a vertical automorphism of a principal bundle  $P \rightarrow X$  which sends  $P^h$  onto  $P^{h'}$  [38, 50]. If the quotient  $G/H$  is diffeomorphic to an Euclidean space (e.g.,  $H$  is a maximal compact subgroup of  $G$ ), all  $H$ -principal subbundles of a  $G$ -principal bundle  $P$  are isomorphic to each other [52].

If a structure group  $G$  of a principal bundle  $P \rightarrow X$  is reducible to its subgroup  $H$ , one can describe matter fields with an exact symmetry group  $H$  as follows. Let  $Y \rightarrow \Sigma$  be a vector bundle associated to the  $H$ -principal bundle  $P_\Sigma$  (22). It is a composite fiber bundle

$$Y \xrightarrow{\pi_{Y\Sigma}} \Sigma \xrightarrow{\pi_{\Sigma X}} X.$$

Let  $h$  be a global section of the fiber bundle  $\Sigma \rightarrow X$ , i.e., a Higgs field. Then the restriction

$$Y_h = h^*Y \tag{24}$$

of the fiber bundle  $Y \rightarrow \Sigma$  to  $h(X) \subset \Sigma$  is a subbundle of the fiber bundle  $Y \rightarrow X$  which is associated to the reduced  $H$ -principal bundle  $P^h$  (23). One can think of sections  $s_h$  of the fiber bundle  $Y_h \rightarrow X$  (24) as being matter fields in the presence of a Higgs field  $h$ . Given a different Higgs field  $h'$ , matter fields in its presence are described by sections of a different fiber bundle  $Y_{h'}$ , which is isomorphic to  $Y_h$  iff the  $H$ -principal bundles  $P^h$  and  $P^{h'}$  are isomorphic. The totality of all the pairs  $(s_h, h)$  of matter and Higgs fields is represented by sections of the fiber bundle  $Y \rightarrow X$  as follows [38].

**Theorem 11.** Since  $Y^h \rightarrow X$  is a subbundle of  $Y \rightarrow X$ , any section  $s_h$  of  $Y^h \rightarrow X$  is a section of  $Y \rightarrow X$  projected onto a section  $h = \pi_{Y\Sigma} \circ s_h$  of the fiber bundle  $\Sigma \rightarrow X$ . Conversely, a section  $s$  of  $Y \rightarrow X$  is a composition  $s = s_\Sigma \circ h$  of a section  $h = \pi_{Y\Sigma} \circ s$  of  $\Sigma \rightarrow X$  and some section  $s_\Sigma$  of the fiber bundle  $Y \rightarrow \Sigma$  whose restriction to the submanifold  $h(X) \subset \Sigma$  is a section  $s_h$  of  $Y_h$ .

Turn now to the properties of gauge fields compatible with spontaneous symmetry breaking. Given a Higgs field  $h$ , the fiber bundle  $Y^h \rightarrow X$  of matter fields is provided with a connection associated to a principal connection on the  $H$ -principal bundle  $P^h$ .

**Theorem 12.** Any principal connection  $A^h$  on a reduced subbundle  $P^h$  of  $P$  gives rise to a principal connection on  $P$ , and yields an associated connection on  $P/H \rightarrow X$  such that the covariant differential  $D_{A^h}h$  of  $h$  vanishes. Conversely, a principal connection  $A$  on  $P$  is projected onto  $P^h$  iff  $D_A h = 0$  [34].

**Theorem 13.** If the Lie algebra  $\mathfrak{g}$  of  $G$  is the direct sum  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  of the Lie algebra  $\mathfrak{h}$  of  $H$  and a subspace  $\mathfrak{m} \subset \mathfrak{g}$  such that  $ad(g)(\mathfrak{m}) \subset \mathfrak{m}$ ,  $g \in H$ , then the pull-back of the

$\mathfrak{H}$ -valued component of any principal connection on  $P$  onto a reduced subbundle  $P^h$  is a principal connection on  $P^h$ . This is the case of so-called reductive  $G$ -structure [21, 38].

Gravitation theory exemplifies classical gauge theory with spontaneously broken symmetries where Dirac spinor fields are matter fields possessing exact Lorentz symmetries and a pseudo-Riemannian metric plays the role of a Higgs field [28, 47, 48, 49].

## References

- [1] V.Alaya, E.Sánchez-Sastre and M.Calixto, Beyond standard gauge theory: extended gauge and diffeomorphism theory and mass generation, *Rep. Math. Phys.* **59** (2007) 83.
- [2] D.Alekseevsky, V.Cortés, D.Chandrashekar and A.Van Proeyen, Polyvector super-Poincaré superalgebras, *Commun. Math. Phys.* **253** (2005) 385.
- [3] G.Arakawa, I.Ichinose, T.Matsui, K.Sakakibara and S.Takashima, Phase structure of a 3D nonlocal  $U(1)$  gauge theory. Deconfinement by gapless matter fields, *Nucl. Phys. B* **732** (2006) 401
- [4] H.Araki, *Mathematical Theory of Quantum Fields* (Oxford Univ. Press, Oxford, 1999).
- [5] C.Bartocci, U.Bruzzo and D.Hernández Ruipérez, *The Geometry of Supermanifolds* (Kluwer, Dordrecht, 1991).
- [6] A.Bélangier and G.Thomas, Positive forms on nuclear  $*$ -algebras and their integral representations, *Can. J. Math.* **42** (1990) 410.
- [7] P.Binétry, G.Girardi and R.Grimm, Supergravity couplings. A geometric formulation, *Phys. Rep.* **343** (2001) 255.
- [8] N.Bogoliubov, A.Logunov, A.Oksak and I.Todorov, *General Principles of Quantum Field Theory* (Kluwer Acad. Publ., Dordrecht, 1990).
- [9] H.Borchers, Algebras of unbounded operators in quantum field theory, *Physica A* **124** (1984) 127.
- [10] O.Bratteli and D.Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vol.1, Second Edit.* (Springer, Berlin, 2002).
- [11] G.Cassinelli, E. de Vito, P.Lahti and A.Levrero, Symmetries of the quantum state space and group representations, *Rev. Math. Phys.* **10** (1998) 893.

- [12] A.Chamseddine, R.Arnouitt and P.Nath, Supergravity unification, *Nucl. Phys. B (Proc. Suppl.)* **101** (2001) 145.
- [13] A.Chamseddine, Spontaneous symmetry breaking for massive spin-2 interacting with gravity, *Phys. Lett. B* **557** (2003) 247.
- [14] T.Clark and S.Love, Nonlinear realizations of supersymmetric AdS space isometries, *Phys. Rev.* **D73** (2006) 025001.
- [15] A.Coleman, Indiced and subduced representations, In: *Group Theory and its Applications*, ed. E.Loeb (Acad. Press, N.Y., 1968) p.57.
- [16] S.Coleman, J.Wess and B.Zumino, Structure of phenomenological Lagrangians, I, II, *Phys. Rev.* **177** (1969) 2239.
- [17] R.D'auria, S.Ferrara, M.Liedo, On the embedding of space-time symmetries into simple superalgebras, *Lett. Math. Phys.* **57** (2001) 123.
- [18] J.Dixmier, *C\*-Algebras* (North-Holland, Amsterdam, 1977).
- [19] M.Dubois-Violette, R.Kerner and J.Madore, Gauge bosons in a noncommutative geometry, *Phys. Lett. B* **217** (1989) 485.
- [20] G.Giachetta, L.Mangiarotti and G.Sardanashvily, *Geometric and Algebraic Topological Methods in Quantum Mechanics* (World Scientific, Singapore, 2005).
- [21] M.Godina and P.Matteucci, Reductive  $G$ -structure and Lie derivatives, *J. Geom. Phys.* **47** (2003) 66.
- [22] F.Gordejuela and J.Masqué, Gauge group and  $G$ -structures, *J. Phys. A* **28** (1995) 497.
- [23] L.Govaerts, Half-integer winding number solutions to the Landau–Ginzburg–Higgs equations and instability of the Abrikosov–Nielsen–Olesen vortex, *J. Phys. A* **34** (2001) 8955.
- [24] R.Haag, *Local Quantum Physics* (Springer, Berlin, 1996).
- [25] S.Horuzhy, *Introduction to Algebraic Quantum Field Theory*, Mathematics and its Applications (Soviet Series) **19** (Kluwer Academic Publ. Group, Dordrecht, 1990).
- [26] S.Igury and M.Castagnino, The formulation of quantum mechanics in terms of nuclear algebras, *Int. J. Theor. Phys.* **38** (1999) 143.

- [27] C.Isham, A.Salam and J.Strathdee, Nonlinear realizations of space-time symmetries. Scalar and tensor gravity, *Ann. Phys.* **62** (1971) 98.
- [28] D.Ivanenko and G.Sardanashvily, The gauge treatment of gravity, *Phys. Rep.* **94** (1983) 1.
- [29] E.Ivanov, S.Krivososov and O.Lechtenfels,  $N = 4$ ,  $d = 1$  supermultiplets from nonlinear realizations of  $D(2, 1; \alpha)$ , *Class. Quant. Grav.* **21** (2004) 1031.
- [30] A.Joseph and A.Solomon, Global and infinitesimal nonlinear chiral transformations, *J. Math. Phys.* **11** (1970) 748.
- [31] Z.Kakushadze and P.Langfelder, Gravitational Higgs mechanism, *Phys. Lett. A* **15** (2000) 2265.
- [32] M.Keyl, About the geometric structure of symmetry breaking, *J. Math. Phys.* **32** (1991) 1065.
- [33] I.Kirsch, A Higgs mechanism for gravity, *Phys. Rev. D* **72** (2005) 024001
- [34] S.Kobayashi and K.Nomizu, *Foundations of Differential Geometry, Vol.1* (Interscience Publ., N.Y., 1963).
- [35] M.Leclerk, The Higgs sector of gravitational gauge theories, *Ann. Phys.* **321** (2006) 708.
- [36] G.Mackey, *Induced Representations of Groups and Quantum mechanics* (W.A.Benjamin, N. Y., 1968).
- [37] S.Mac Lane, *Homology* (Springer, Berlin, 1967).
- [38] L.Mangiarotti and G.Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore, 2000).
- [39] B.Melić, K.Passek-Kumerički, J.Trampetić, P.Schupp and M.Wohlgenannt, The standard model on noncommutative space-time: electroweak currents and the Higgs sector, *Eur. Phys. J. C* **42** (2005) 483.
- [40] R.Nevzorov, K.Ter-Martirosyan and M.Trusov, Higgs bosons in the symplectic SUSY models, *Phys. Atomic Nuclei* **65** (2002) 285.
- [41] P.Van Nieuwenhuizen, Supergravity, *Phys. Rep.* **68** (1981) 189.

- [42] L.Nikolova and V.Rizov, Geometrical approach to the reduction of gauge theories with spontaneous broken symmetries, *Rep. Math. Phys.* **20** (1984) 287.
- [43] H.Nilles, Supersymmetry, supergravity and particle physics, *Phys. Rep.* **110** (1984) 1.
- [44] T.Ohl and J.Reuter, Testing the noncommutative standard model at a future photon collider, *Phys. Rev. D* **70** (2004) 076007.
- [45] R.Powers and S.Sakai, Unbounded derivations in operator algebras, *J. Funct. Anal.* **19** (1975) 81.
- [46] G.Sardanashvily, On the geometry of spontaneous symmetry breaking, *J. Math. Phys.* **33** (1992) 1546.
- [47] G.Sardanashvily, Covariant spin structure, *J. Math. Phys.* **39** (1998) 4874.
- [48] G.Sardanashvily, Classical gauge theory of gravity, *Theor. Math. Phys.* **132** (2002) 1163.
- [49] G.Sardanashvily, Gauge gravitation theory from geometric viewpoint, *Int. J. Geom. Methods Mod. Phys.* **3** (2006) N1, v-xx.
- [50] G.Sardanashvily, Geometry of classical Higss field, *Int. J. Geom. Methods Mod. Phys.* **3** (2006) 139.
- [51] K.Schmüdgen, *Unbounded Operator Algebras and Representation Theory* (Birkhäuser, Berlin, 1990).
- [52] N.Steenrod, *The Topology of Fibre Bundles* (Princeton Univ. Press, Princeton, 1972).
- [53] A.Tiemblo and R.Tresguerres, Gauge theories of gravity: the nonlinear framework, *Res. Devel. Phys.* **5** (2004) 1255.
- [54] F.Trevers, *Topological Vector Spaces, Distributions and Kernels* (Academic Press, New York, 1967).
- [55] V.Varadarjan, *Geometry of Quantum Theory* (Springer, Berlin, 1985).
- [56] J.Wess and J.Bagger, *Supersymmetry and Supergravity*, Princeton Series in Physics (Princeton Univ. Press, Princeton, 1992).
- [57] A.Wightman and L.Garding, Fields as operator valued distributions in relativistic quantum field theory, *Arkiv Fys.* **28** (1964) 129.
- [58] Yu.Zinoviev, Equivalence of Euclidean and Wightman field theories, *Commun. Math. Phys.* **174** (1995) 1.